Midterm Exam Calculus 2

21 March 2024, 18:30-20:30

university of groningen

The exam consists of 4 problems. You have 120 minutes to answer the questions. You can achieve 100 points which includes a bonus of 10 points. Calculators, books and notes are not permitted.

1. [**8+7+5 = 20 Points**]
   Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases} .$$

   (a) Use the definition of partial derivatives to calculate $f_x(0,0)$ and $f_y(0,0)$.

   (b) Let $a \in \mathbb{R}$ with $a \neq 0$, and let $\mathbf{r}(t) = (t, at)$. Show that the composite function $f \circ \mathbf{r} : \mathbb{R} \to \mathbb{R}$, $t \mapsto f(\mathbf{r}(t))$ is differentiable at $t = 0$.

   (c) Compute $\nabla f(0,0) \cdot \mathbf{r}'(0)$. Reconcile this result with your result in part (b) to conclude on the differentiability of f at $(x,y) = (0,0)$.

2. [**12+8 = 20 Points**]
   Let C be the curve parametrized by $\mathbf{r} : [0, 2\pi] \to \mathbb{R}^3$, $t \mapsto \mathbf{r}(t)$ with

$$\mathbf{r}(t) = \mathrm{e}^t \cos t \,\mathbf{i} + \mathrm{e}^t \sin t \,\mathbf{j} + \mathrm{e}^t \,\mathbf{k}.$$

   (a) Find the length of the curve C and its parametrization by arc length.

   (b) For each point on C, compute the curvature of C at this point.

3. [**5+10+10 = 25 Points**]
   Let S be the ellipsoid in $\mathbb{R}^3$ defined by

$$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 3$$

   which contains the point $(x_0, y_0, z_0) = (1, 2, 3)$.

   (a) Compute the tangent plane of S at the point (x_0, y_0, z_0).

   (b) Use the Implicit Function Theorem to show that near the point (x_0, y_0, z_0) the ellipsoid S is locally given as the graph of a function over the (x,y) plane, i.e. there is a function $f : (x,y) \mapsto f(x,y)$ such that near (x_0, y_0, z_0) the ellipsoid is locally given by $z = f(x,y)$. Compute the partial derivatives f_x and f_y at (x_0, y_0) and show that the graph of the linearization of f at (x_0, y_0) agrees with the tangent plane found in part (a).

— please turn over —

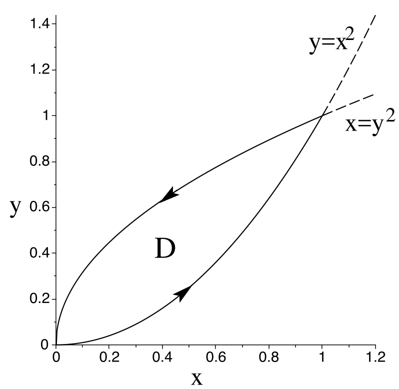
(c) For a point $P = (x, y, z)$ in S, there is a box inscribed in S with corners (x, y, z), $(x, y, -z)$, $(x, -y, -z)$, $(x, -y, z)$, $(-x, y, z)$, $(-x, y, -z)$, $(-x, -y, -z)$ and $(-x, -y, z)$. Use the method of Lagrange multipliers to determine the box with largest possible volume.

4. [**25 Points**]

Let D be the region in the first quadrant of $\mathbb{R}^2$ enclosed by the parabolas $y = x^2$ and $x = y^2$ as shown in the figure below. For the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j} = xy\,\mathbf{i} + y^2\,\mathbf{j}$, compute both sides of the equation

$$\iint_D \left(\frac{\partial}{\partial x} Q - \frac{\partial}{\partial y} P \right) \mathrm{d}A = \oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y,$$

where C is the piecewise smooth curve that forms the boundary of D with the orientation indicated by the arrows in the figure.



Solutions

1. (a) Following the definition, the partial derivative of f with respect to x at $(x, y) = (0, 0)$ is

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^2\cdot 0}{h^2+0^2} - 0}{h} = \lim_{h\to 0} 0 = 0.$$

Similarly

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{0^2\cdot h}{0^2+h^2} - 0}{h} = \lim_{h\to 0} 0 = 0.$$

(b) Let $g = f \circ \mathbf{r}$. Then

$$g(t) = \begin{cases} \frac{at^3}{t^2+a^2t^2} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}.$$

For the differentiability of g at $t = 0$ consider for $h \neq 0$, the difference quotient

$$\frac{g(h) - g(0)}{h} = \frac{\frac{ah^3}{h^2+a^2h^2} - 0}{h} = \frac{ah^3}{h^3 + a^2h^3} = \frac{a}{1+a^2}.$$

As the difference quotient has a limit for $h \to 0$ we conclude that g is differentiable at $t = 0$ and the derivative is $g'(0) = \frac{a}{1+a^2}$.

(c) From part (a) we have $\nabla f(0,0) = (0,0)$. We have $\mathbf{r}'(0) = (1, a)$. So $\nabla f(0,0) \cdot \mathbf{r}'(0) = 0$. If f would be differentiable at $(x, y) = (0, 0)$ then by the Chain Rule the derivative of $f \circ \mathbf{r}$ at $t = 0$ would be $\nabla f(0,0) \cdot \mathbf{r}'(0) = 0$ which does not agree with the result in part (b). We conclude that f is not differentiable at $(x, y) = (0, 0)$.

2. (a) The arc length is defined as

$$s(t) = \int_0^t |\mathbf{r}'(\tau)|\,\mathrm{d}\tau\,.$$

We have $\mathbf{r}'(t) = \mathrm{e}^t(\cos t - \sin t)\,\mathbf{i} + \mathrm{e}^t(\sin t + \cos t)\,\mathbf{j} + \mathrm{e}^t\,\mathbf{k}$, and hence

$$|\mathbf{r}'(t)| = \mathrm{e}^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1} = \mathrm{e}^t\sqrt{2\cos^2 t + 2\sin^2 t + 1} = \mathrm{e}^t\sqrt{3}.$$

Hence

$$s(t) = \int_0^t \mathrm{e}^\tau\sqrt{3}\,\mathrm{d}\tau = (\mathrm{e}^t - 1)\sqrt{3}.$$

Solving for t gives

$$t(s) = \ln(\frac{s}{\sqrt{3}} + 1).$$

So the parametrization of C by arc length is given by

$$\begin{aligned}\tilde{\mathbf{r}}(s) &= \mathbf{r}(t(s)) = \mathrm{e}^{t(s)}\cos t(s)\,\mathbf{i} + \mathrm{e}^{t(s)}\sin t(s)\,\mathbf{j} + \mathrm{e}^{t(s)}\,\mathbf{k} \\ &= (\frac{s}{\sqrt{3}} + 1)(\cos\ln(\frac{s}{\sqrt{3}} + 1)\,\mathbf{i} + \sin\ln(\frac{s}{\sqrt{3}} + 1)\,\mathbf{j} + \mathbf{k})\end{aligned}$$

where $0 \le s \le (\mathrm{e}^{2\pi} - 1)\sqrt{3}$.

(b) The curvature κ is defined as

$$\kappa = \left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}s}\right|,$$

where $\mathbf{T}$ is the unit tangent vector. By the chain rule

$$\kappa = \frac{1}{|\mathbf{r}'(t)|}\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right|,$$

From part (a) we get

$$\mathbf{T} = \frac{1}{|\mathbf{r}'(t)|}\mathbf{r}'(t) = \frac{1}{\sqrt{3}}\big((\cos t - \sin t)\,\mathbf{i} + (\sin t + \cos t)\,\mathbf{j} + \mathbf{k}\big)$$

which gives

$$\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t} = \frac{1}{\sqrt{3}}\big((-\sin t - \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}\big)$$

and hence by a similar computation as in part (a)

$$\left|\frac{\mathrm{d}\mathbf{T}}{\mathrm{d}t}\right| = \left|\frac{1}{\sqrt{3}}\big((-\sin t - \cos t)\,\mathbf{i} + (\cos t - \sin t)\,\mathbf{j}\big)\right| = \frac{\sqrt{2}}{\sqrt{3}}.$$

The curvature of C at $\mathbf{r}(t)$ is thus

$$\kappa = \frac{1}{\mathrm{e}^t\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{3}\mathrm{e}^{-t}.$$

3. (a) The ellipsoid S is given by the zero-level set of the function $F(x,y,z) = x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 3$. We can hence find a normal vector of the tangent plane of S at (x_0,y_0,z_0) from $\nabla F(x_0,y_0,z_0) = 2x_0\mathbf{i} + \frac{1}{2}y_0\mathbf{j} + \frac{2}{9}z_0\mathbf{k} = 2\mathbf{i} + 1\mathbf{j} + \frac{2}{3}\mathbf{k}$. The tangent plane is hence given by $\nabla F(x_0,y_0,z_0)\cdot(x-x_0, y-y_0, z-z_0) = 0$, i.e.

$$2(x-1) + (y-2) + \frac{2}{3}(z-3) = 0$$

or equivalently,

$$6x + 3y + 2z = 18.$$

(b) Using the fact that S is given by the zero-level set of the function F defined in part (a) the local existence of the function f follows from the Implicit Function Theorem if we can show that $\frac{\partial F}{\partial z}(x_0,y_0,z_0) \neq 0$. The latter follows from $\frac{\partial F}{\partial z}(x_0,y_0,z_0) = \frac{2}{9}z_0 = \frac{2}{3}$. The Implicit Function Theorem gives

$$f_x(x_0,y_0) = -\frac{F_x(x_0,y_0,z_0)}{.F_z(x_0,y_0,z_0)} = -\frac{2x_0}{\frac{2}{9}z_0} = -\frac{2}{\frac{2}{3}} = -3$$

and

$$f_y(x_0,y_0) = -\frac{F_y(x_0,y_0,z_0)}{F_z(x_0,y_0,z_0)} = -\frac{\frac{1}{2}y_0}{\frac{2}{9}z_0} = -\frac{1}{\frac{2}{3}} = -\frac{3}{2}.$$

The linearization of f at (x_0,y_0) is given by

$$\begin{aligned} L(x,y) &= f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) \\ &= 3 - 3(x-1) - \frac{3}{2}(y-2) \\ &= 9 - 3x - \frac{3}{2}y. \end{aligned}$$

The graph of the linearization is given by the equation $z = L(x, y)$ which agrees with the tangent plane found in part (a).

(c) The volume of the box is given by $V(x, y, z) = 8xyz$. It follows from the Theorem on Lagrange Multipliers that at a critical point $(x, y, z) \in S$ of V restricted to S there is a $\lambda \in \mathbb{R}$ such that $\nabla V(x, y, z) = \lambda \nabla F(x, y, z)$ where we use that the constraint S is given by the zero-level set of the function F defined in part (a). In order to find the critical points we have to solve the set of equations

$$\begin{aligned} V_x(x, y, z) &= \lambda F_x(x, y, z), \\ V_y(x, y, z) &= \lambda F_y(x, y, z), \\ V_z(x, y, z) &= \lambda F_z(x, y, z), \\ F(x, y, z) &= 0. \end{aligned}$$

for x, y, z and λ. This gives

$$\begin{aligned} 8yz &= \lambda 2x, \\ 8xz &= \lambda \frac{1}{2} y, \\ 8xy &= \lambda \frac{2}{9} z, \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} &= 3, \end{aligned}$$

As for $x = 0$, $y = 0$ or $z = 0$, $V(x, y, z) = 0$ we can assume $x, y, z \neq 0$. We then get

$$\begin{aligned} 4\frac{yz}{x} &= \lambda, \\ 16\frac{xz}{y} &= \lambda, \\ 36\frac{xy}{z} &= \lambda, \\ x^2 + \frac{y^2}{4} + \frac{z^2}{9} &= 3. \end{aligned}$$

Equating the left sides of the first and the second equality gives $y = 2x$. Equating the left sides of the first and the third equality gives $z = 3x$. This allows one to eliminate y and z in the last equality to get $3x^2 = 3$ which gives $x = 1$ (or $x = -1$ which we can discard by symmetry). So we get $y = 2x = 2$ and $z = 3x = 3$. The largest volume is hence $V = 8 \cdot 1 \cdot 2 \cdot 3 = 48$.

4. (a) We start with the computation of the left hand side. We have

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - x = -x.$$

Hence

$$\begin{aligned} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}A &= \int_0^1 \int_{x^2}^{\sqrt{x}} -x \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 -x(\sqrt{x} - x^2) \, \mathrm{d}x \\ &= \int_0^1 (-x^{3/2} + x^3) \, \mathrm{d}x = -\frac{2}{5} x^{5/2} + \frac{1}{4} x^4 \Big|_0^1 \\ &= -\frac{2}{5} + \frac{1}{4} = \frac{5 - 8}{20} = -\frac{3}{20}. \end{aligned}$$

We now compute the right hand side of the equation. We have $C = C_1 \cup C_2$ where C_1 corresponds to the part of the parabola $y = x^2$ which has parametrization $\mathbf{r}_1(t) = (t, t^2)$, $0 \le t \le 1$. The tangent vector corresponding to the parametrization $\mathbf{r}_1$ gives the desired orientation shown in figure. The part C_2 corresponds to the parabola $x = y^2$ which can be parametrized by $\mathbf{r}_2(t) = \big((1-t)^2, 1-t\big)$ with $0 \le t \le 1$. The tangent vector associated with $\mathbf{r}_2$ gives the desired orientation on C_2 shown in the figure.. Using $\mathbf{r}_1'(t) = (1, 2t)$ and $\mathbf{r}_2'(t) = (-2(1-t), -1)$, we get

$$\begin{aligned}
\oint_C P\mathrm{d}x + Q\mathrm{d}y &= \int_0^1 \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{r}_1'(t)\,\mathrm{d}t + \int_0^1 \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{r}_2'(t)\,\mathrm{d}t \\
&= \int_0^1 (t \cdot t^2, t^4) \cdot (1, 2t)\,\mathrm{d}t + \int_0^1 \big((1-t)^2(1-t), (1-t)^2\big) \cdot (-2(1-t), -1)\,\mathrm{d}t \\
&= \int_0^1 (t^3 + 2t^5)\,\mathrm{d}t + \int_0^1 \big(-2(1-t)^4 - (1-t)^3\big)\,\mathrm{d}t \\
&= \frac{1}{4} + \frac{2}{6} + \int_0^1 \big(2s^4 + s^2\big)\,\mathrm{d}s \\
&= \frac{1}{4} + \frac{1}{3} + \big(\frac{2}{5}s^5 + \frac{1}{3}s^3\big)\Big|_1^0 \\
&= \frac{1}{4} + \frac{1}{3} - \frac{2}{5} - \frac{1}{3} = \frac{5-8}{20} = -\frac{3}{20}.
\end{aligned}$$